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Identification of the Deterministic Part of MIMO State Space Models given in Innovations Form from Input–Output Data*

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The problem of Linear Multivariable State Space model identification from input–output data can under the presence of process- and measurement noise be solved in a non-iterative way when incorporating instrumental variables constructed from both input and output sequences in the recently developed class of multivariable output-error state space model class of subspace model identification schemes.

Key Words—State space; system identification; linear systems; linear algebra.

Abstract—In this paper we describe two algorithms to identify a linear, time-invariant, finite dimensional state space model from input–output data. The system to be identified is assumed to be excited by a measurable input and an unknown *process noise* and the measurements are disturbed by unknown *measurement noise*. Both noise sequences are discrete zero-mean white noise. The first algorithm gives consistent estimates only for the case where the input also is zero-mean white noise, while the same result is obtained with the second algorithm without this constraint. For the special case where the input signal is discrete zero-mean white noise, it is explicitly shown that this second algorithm is a special case of the recently developed Multivariable Output-Error State Space (MOESP) class of algorithms based on instrumental variables. The usefulness of the presented schemes is highlighted in a realistic simulation study.

1. INTRODUCTION

The identification of multiple-input, multiple-output (MIMO) linear time-invariant state space models from input–output measurements is a problem of central importance in system analysis, design and control. In general terms, it can be viewed as the problem of finding a mapping between the available input–output data sequences and unknown parameters in a user-defined class of models, e.g. state space models.

A particular class of solutions, discussed in Ljung (1987) or Söderström and Stoica (1989), tackle versions of this general problem in a direct way by using iterative optimization schemes. The major drawbacks of this direct approach are the difficulty of model class selection, the difficulty of parametrization of a MIMO state space model, and the possibility of parameter divergence and local minima in the numerical optimization.

An indirect approach is the subspace model identification (SMI) approach (Verhaegen and Deprettere, 1991). Here the intermediate key step is the approximation of a structured subspace from spaces defined by Hankel matrices constructed from the input–output data. For example in the MOESP family of algorithms (Verhaegen, 1993a), this structured subspace is the extended observability matrix. The structure of this subspace then allows us to approximate certain parts of the state space representation describing the input–output transfer of the given system.

In (Verhaegen and Dewilde, 1992a), a very general identification problem was formulated and in (Verhaegen, 1993a) a realistic version of this problem was solved. The generality of this problem stemmed from the fact that the errors, which were assumed to be summed at the output, could have an arbitrary statistical colour. In this paper, a more restrictive class of perturbations (at the output) is considered. Namely, we assume that the finite dimensional linear time-invariant (FDLTI) system, shown in Fig. 1, is given by a state space model in an

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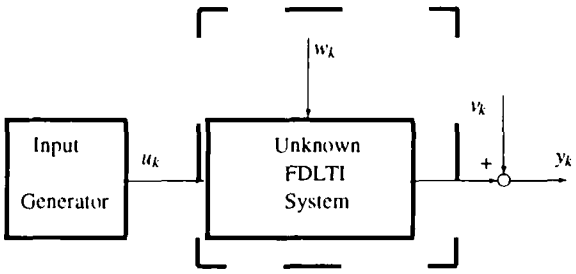

FIG. 1. Block schematic view of a realistic system open-loop identification set-up.

innovation form (Ljung, 1987). Thus the sequences $\{u_k\}$ and $\{w_k\}$ act as a joint input sequence to a linear system given in a standard state space form. The difference between the sequences $\{u_k\}$ and $\{w_k\}$ is that u_k is generated and therefore precisely known, while w_k is an unknown white noise sequence. Both sequences are assumed to be statistically independent.

For this class of systems, we treat the following identification problem.

—Given the input–output sequences $\{u_k\}$ and $\{y_k\}$ and assuming that the additive perturbation v_k shown in Fig. 1 is also a zero-mean white noise sequence, which is statistically independent from the input u_k, then the task is to find a consistent estimate of the FDLTI state space model that models the (deterministic) transfer between the input u_k and the output y_k.

Different solutions have been proposed to this problem in the SMI context. First we have the work of Larimore (1990) on canonical variate analysis. This work is a continuation of the pioneering activities initiated by Akaike (1975) and treats the problem mainly in a statistical setting. Second we have the work of Van Overschee (Overschee and De Moor, 1992). Although the conceptual nature of these two approaches is similar, the derivation of the solution in Overschee and De Moor (1992) is now based purely on standard linear algebra.

In this paper we treat the above identification problem in the MOESP framework, initiated in Verhaegen (1990), but later extended in the series of papers by Verhaegen (1993a), and Verhaegen and Dewilde (1992a, b). The basis of deriving solutions in the MOESP framework is again linear algebra. The main differences between the presented solution and those mentioned in the previous paragraph are two-fold. First, conceptually, the key subspace in our approach is the extended observability matrix instead of the state sequence, which is defined as an intersection between past and future input–output data Hankel matrices. The second main difference is algorithmical. The solution presented in Larimore (1990) is based on the canonical correlation analysis, while that in Overschee and De Moor (1992) calculates the required state sequence from an explicit projection of the future output data Hankel matrix onto the compound matrix of past and future input data Hankel matrix and past output data Hankel matrix. The schemes derived in this paper preserve the main characteristics of the MOESP class of algorithms. These are, first, the compression of a compound matrix of input and output Hankel matrices into a lower triangular matrix by means of orthogonal transformations. The latter need not to be stored explicitly. Second, the column space of specific submatrices of the resulting lower triangular factor approximates the column space of the extended observability matrix in a consistent way. This common nature of the algorithms deviced will contribute to their being understood and to the simplification of their implementation.

Although the model class treated in the above identification problem is not of the output-error class, it will be shown in this paper that the solution presented is another variant of the ordinary MOESP scheme extended by means of instrumental variables (Verhaegen, 1993a).

The paper is organized as follows. In Section 2 some basic notation, the model and data representation used throughout the paper, is described. The approximation of the column space of the extended observability matrix is then treated in Section 3. In particular we analyze the consistency of the proposed approximations in this section. These results are then used in Section 4 to derive two identification schemes that allow us to solve the above identification problem in a consistent way. The relationship of one of the derived schemes with the ordinary MOESP scheme extended by instrumental variables is highlighted in Section 5 and Section 6 illustrates the usefulness of the presented solution by means of one realistic simulation study. Finally, we present some concluding remarks in Section 7.

2. MODEL DESCRIPTION AND NOTATIONAL PRELIMINARIES

2.1. *Some basic notations*

In this section, we define some notation frequently used in this paper.

- Matrix partitioning: an initial method of indicating the partitioning of a matrix or a vector is illustrated by the following example.

Example 1. Let $A \in R^{(m+\ell)\times N}$, $m \leq N$, $\ell \geq 0$,

then the following representation of A:

$$A = \begin{array}{c} \\ m \\ \ell \end{array} \begin{array}{c} \begin{array}{cc} m & N-m \end{array} \\ \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \end{array}$$

indicates the partitioning of A into respectively $A_{11} \in R^{m \times m}$, $A_{12} \in R^{m \times (N-m)}$, $A_{21} \in R^{\ell \times m}$ and $A_{22} \in R^{\ell \times (N-m)}$.

A second method of indicating the partitioning is consistent with the notation used in the MATLAB package (Moler *et al.*, 1987).

- The rank of a matrix: the rank of a matrix A, defined, for example, in Golub and Van Loan (1989), is denoted by $\rho(A)$.
- The RQ factorization: the RQ factorization of a matrix $A \in R^{m \times N}$ is a factorization of this matrix into a lower triangular matrix $R \in R^{m \times N}$ and a square orthogonal matrix $Q \in R^{N \times N}$, such that:

$$A = RQ.$$

- The quadruple (A_T, B_T, C_T, D): defines a quadruple of system matrices that are equal up to a similarity transformation T to the quadruple of system matrices (A, B, C, D), that is:

$$(A_T, B_T, C_T, D) = (TAT^{-1}, TN, CT^{-1}, D).$$

- I_ℓ: this denotes the identity matrix of order ℓ.

2.2. *The innovations form of the state space description*

The FDLTI system in Fig. 1 is assumed to be represented by:

$$x_{k+1} = Ax_k + Bu_k + Fw_k, \tag{1}$$

$$y_k = Cx_k + Du_k + Gw_k + v_k, \tag{2}$$

where $x_k \in R^n$, $u_k \in R^{m_1}$, y_k, $v_k \in R^\ell$ and $w_k \in R^{m_2}$. The unknown system matrices (A, B, C, D) and (F, G) have appropriate dimensions. The process noise w_k and the measurement noise v_k are zero-mean white noise sequences, statistically independent of the input u_k. They satisfy:

$$E\left[\begin{pmatrix} w_k \\ v_k \end{pmatrix}(w_j^T v_j^T)\right] = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \quad \text{for } k = j \tag{3}$$
$$= 0 \qquad \text{for } k \neq j,$$

where $E[.]$ represents the mathematical expectation operator.

There is a whole class of systems of the form (1–3) that have an output with the same second order statistics (Anderson and Moore, 1979). Strictly speaking, the innovations form in such a class has the following form (Ljung, 1987, p. 87):

$$x_{k+1} = Ax_k + Bu_k + Ke_k$$
$$y_k = Cx_k + Du_k + e_k,$$

where K is the Kalman gain and e_k the innovation. In this paper, we assume the systems to be identified to be modelled by equations (1–3).

For such systems, it is assumed that the pair (A, C) is observable and the pair $(A, (B\,FQ^{\perp}))$ is controllable. Further, throughout this paper we assume the system to be asymptotically stable, the input sequence u_k to be known exactly and the sequence y_k to be the measured output sequence of the FDLTI system.

2.3. *Data representation*

We will use the ergodic–algebraic framework, such as proposed in Verhaegen and Dewilde (1992a), to analyze the occurring stochastic processes. Thus, we assume the signals in the identification problem to be (finite segments of) realizations of ergodic–stochastic processes. That is, for $N \to \infty$ there are ergodic–stochastic processes $\mathbf{u_j} \in R^m$ and $\mathbf{v_k} \in R^\ell$ such that:

$$(u_j\, u_{j+1} \cdots u_{N+j-1}) \quad \text{and} \quad (v_k\, v_{k+1} \cdots v_{N+k-1}) \tag{4}$$

are realizations of $\mathbf{u_j}$ and $\mathbf{v_k}$, respectively and the following (or similar) expression(s) holds:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} u_{j+i-1} v_{k+i-1}^T = E[\mathbf{u_j v_k^T}]. \tag{5}$$

We adopt the notation, in bold, to represent the stochastic process. An alternative way of expressing the above limit is:

$$\frac{1}{N} \sum_{i=1}^{N} u_{j+i-1} v_{k+i-1}^T + O_N(\varepsilon) = E[\mathbf{u_j v_k^T}], \tag{6}$$

where $O_N(\varepsilon)$ is a bounded matrix of appropriate dimensions of norm ε which vanishes for $N \to \infty$.

In the derivation of the identification schemes in this paper, the organization of the data in structured matrices plays a crucial role. Three types of structured matrices are used. First, the available data sequences u_k, y_k and process noise and measurement noise are collected in Hankel matrices. For example, for the input sequence u_k, we define the Hankel matrix $U_{j,s,N}$ as:

$$U_{j,s,N} = \begin{pmatrix} u_j & u_{j+1} & \cdots & u_{j+N-1} \\ u_{j+1} & u_{j+2} & \cdots & u_{j+N} \\ \vdots & & \ddots & \\ u_{j+s-1} & u_{j+s} & \cdots & u_{j+N+s-2} \end{pmatrix}.$$

Similarly, we can construct the Hankel matrices $Y_{j,s,N}$, $W_{j,s,N}$ and $V_{j,s,N}$. Second, we

have the Toeplitz matrices H_s and E_s defined from the system matrices (A, B, C, D) and (F, G), respectively as:

$$H_s = \begin{pmatrix} D & 0 & & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & & 0 \\ \vdots & & & \ddots & \vdots \\ CA^{s-2}B & & & \cdots & D \end{pmatrix}$$

$$E_s = \begin{pmatrix} G & 0 & & \cdots & 0 \\ CF & G & 0 & \cdots & 0 \\ CAF & CF & G & & 0 \\ \vdots & & & \ddots & \vdots \\ CA^{s-2}F & & & \cdots & G \end{pmatrix}$$

Finally, the state sequence $(x_j x_{j+1} \cdots x_{j+N-1})$ gives rise to the matrix $X_{j,N}$ and the extended observability matrix Γ_s is defined as:

$$\Gamma_s = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{s-1} \end{pmatrix}.$$

From the state space model representation (1–2), the derivation of the following algebraic relationship between these three types of structured matrices is straightforward:

$$Y_{j,s,N} = \Gamma_s X_{j,N} + H_s U_{j,s,N} + E_s W_{j,s,N} + V_{j,s,N}. \quad (7)$$

Throughout this paper, we often treat zero-mean white noise sequences. A formal definition is given next.

Definition 1. A sequence $u_k \in R^m$ is a zero-mean white noise sequence, if it has mean zero and if it satisfies the following condition:

$$E[\mathbf{u_k u_j^T}] = \sigma_u^2 I_m \quad \text{for } k = j$$
$$= 0 \quad \text{for } k \neq j. \quad (8)$$

Finally, in this paper, we assume that the input signal u_k is chosen such that (i) all controllable modes of the pair (A, B) have been excited and (ii) $\rho(U_{1,2s,N}) = 2sm_1$. Such an input is referred to as a sufficiently persistently exciting input. More precise statements on the necessary order of persistency of excitation can be made. Such statements depend on the nature of the input signal used and the order of the system. However, since these precise statements will follow from a similar analysis such as made in Verhaegen and Dewilde (1992a) for the class of SMI schemes treated in that paper, and since most of the signals used in an identification context are sufficiently persistently exciting, we refrain from making such an analysis in this paper.

3. APPROXIMATING THE EXTENDED OBSERVABILITY MATRIX Γ_s

In Verhaegen and Dewilde (1992a) and Verhaegen (1993a), it is shown that useful information about the extended observability matrix Γ_s can be derived from a simple RQ factorization of a compound matrix constructed from the Hankel matrices $U_{j,s,N}$ and $Y_{j,s,N}$. Here, we first consider the following RQ factorization:

$$\begin{pmatrix} U_{1,s,N} \\ U_{s+1,s,N} \\ Y_{1,s,N} \\ Y_{s+1,s,N} \end{pmatrix} = \begin{pmatrix} R_{11}^N & & & \\ R_{21}^N & R_{22}^N & & \\ R_{31}^N & R_{32}^N & R_{33}^N & \\ R_{41}^N & R_{42}^N & R_{43}^N & R_{44}^N \end{pmatrix} \times \begin{pmatrix} Q_1^N \\ Q_2^N \\ Q_3^N \\ Q_4^N \end{pmatrix}. \quad (9)$$

This relationship is again purely algebraic.

The relevance of some matrices in the R factor in equation (9) is revealed in Theorem 1. In the proof of this theorem, we make use of the following lemmas.

Lemma 1. Let the input u_k to the system (1–2) be discrete zero-mean white noise, and let the RQ factorization in equation (9) be given, then:

$$\lim_{N \to \infty} \frac{1}{\sqrt{N}} R_{21}^N = 0.$$

Proof. See Appendix A.1.

The white noise property of the process noise w_k and measurement noise v_k gives rise to the following two lemmas.

Lemma 2. Let the process noise w_k of the system (1–2) be discrete zero-mean white noise, and let it be independent of the initial state x_0, then:

$$E[\mathbf{x_k w_j^T}] = 0 \quad \text{for } j \geq k \geq 0.$$

Proof. A proof of this lemma is given in Verhaegen and Dewilde (1992a) (Lemma 2).

Lemma 3. Let the process noise w_k and the measurement noise v_k of the system (1–2) be discrete zero-mean white noise, statistically independent of the input u_j for all k, j and of the initial state x_0, let the input u_j be sufficiently persistently exciting, and let the RQ factoriza-

tion in equation (9) be given, then:

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} W_{s+1,s,N}(Q_3^N)^T = 0$$

and

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} V_{s+1,s,N}(Q_3^N)^T = 0.$$

Proof. See Appendix A.2.

Remark 1. The lemma also holds when the two block matrices $U_{1,s,N}$ and $U_{s+1,s,N}$ in equation (8) are interchanged. □

We can now state the first theorem of the paper based on these three lemmas.

Thoerem 1. Let the process noise w_k and the measurement noise v_k of the system (1–2) be discrete zero-mean white noise, independent of the input u_j for all k, j and of the initial state x_0, let the input u_k to the system (1–2) be discrete zero-mean white noise, and let the RQ factorization in equation (9) be given, then:

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} Y_{s+1,s,N}(Q_1^N)^T = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \Gamma_s X_{s+1,N}(Q_1^N)^T, \quad (10)$$

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} Y_{s+1,s,N}(Q_3^N)^T = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \Gamma_s X_{s+1,N}(Q_3^N)^T. \quad (11)$$

Proof. See Appendix A.3.

Using the representation in equation (6) and the RQ factorization in equation (9), the limits in equations (10) and (11) can alternatively be denoted as:

$$R_{41}^N = \Gamma_s X_{s+1,N}(Q_1^N)^T + O_N(\varepsilon), \quad (12)$$

$$R_{43}^N = \Gamma_s X_{s+1,N}(Q_3^N)^T + O_N(\varepsilon). \quad (13)$$

Let the SVD of $[R_{41}^N \quad R_{43}^N]$ be given as:

$$[R_{41}^N \quad R_{43}^N] = \ell s(\underset{n}{U_n} \mid \underset{\ell s-n}{U_{n^\perp}}) \times \left(\begin{array}{c|c} \underset{n}{S_n} & \underset{\ell s-n}{0} \\ \hline 0 & S_2 \end{array}\right) \begin{pmatrix} V_n^T \\ (V_n^\perp)^T \end{pmatrix}, \quad (14)$$

then the column space of U_n approximates that of Γ_s. Let T be a non-singular $n \times n$ matrix, then this approximation can be denoted by:

$$\Gamma_s \approx U_n T. \quad (15)$$

Theorem 1 is only valid for the case where the input is zero-mean discrete white noise. For an arbitrary (persistently exciting) input signal, a biased estimate will result. In an experimental analysis, see Section 6.1, we observed that also for small sized data batches, i.e. small N, the obtained estimates are biased even when the input was taken as a short segment of a realization of a white noise sequence.

To overcome these deficiencies, we consider a second RQ factorization in the next Theorem.

Theorem 2. Let the process noise w_k and the measurement noise v_k of the system (1–2) be discrete zero-mean white noise, independent of the input u_j for all k, j and of the initial state x_0, let the input u_k to the system (1–2) be sufficiently persistently exciting and let the following RQ factorization be given:

$$\begin{pmatrix} U_{s+1,s,N} \\ U_{1,s,N} \\ Y_{1,s,N} \\ Y_{s+1,s,N} \end{pmatrix} = \begin{pmatrix} R_{11}^{N'} & & & \\ R_{21}^{N'} & R_{22}^{N'} & & \\ R_{31}^{N'} & R_{32}^{N'} & R_{33}^{N'} & \\ R_{41}^{N'} & R_{42}^{N'} & R_{43}^{N'} & R_{44}^{N'} \end{pmatrix} \times \begin{pmatrix} Q_1^{N'} \\ Q_2^{N'} \\ Q_3^{N'} \\ Q_4^{N'} \end{pmatrix}, \quad (16)$$

then:

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} Y_{s+1,s,N}(Q_2^{N'})^T = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \Gamma_s X_{s+1,N}(Q_2^{N'})^T, \quad (17)$$

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} Y_{s+1,s,N}(Q_3^{N'})^T = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \Gamma_s X_{s+1,N}(Q_3^{N'})^T. \quad (18)$$

Proof. See Appendix A.4.

As a consequence of Theorem 2, let the SVD of $[R_{42}^{N'} \quad R_{43}^{N'}]$ be given as:

$$[R_{42}^{N'} \quad R_{43}^{N'}] = \ell s(\underset{n}{U_{n'}} \mid \underset{\ell s-n}{U_{n^\perp}}) \left(\begin{array}{c|c} \underset{n}{S_{n'}} & \underset{\ell s-n}{0} \\ \hline 0 & S_2' \end{array}\right) \begin{pmatrix} (V_n')^T \\ (V_n^{\perp'})^T \end{pmatrix} \quad (19)$$

then the column space of U_n' approximates that of Γ_s in a consistent way.

4. APPROXIMATING THE QUADRUPLE (A_T, B_T, C_T, D)

4.1. *Approximating the pair* (A_T, C_T)

Theorems 1 and 2 of the previous section provide a consistent estimate of the column space of Γ_s. When this column space is exactly known, we can calculate, as in Verhaegen and

Dewilde (1992a), the matrix A_T and C_T. To see this, let equality hold in equation (15), then by the special structure of Γ_s, the matrix A_T and C_T satisfy (Verhaegen and Dewilde, 1992a):

$$U_n(1:\ell(s-1),:)A_T = U_n(\ell+1:\ell s,:),$$

and

$$C_T = U_n(1:\ell,:), \tag{21}$$

when N is finite, we use the approximations of the column space as obtained by the SVDs in equation (14) or in equation (19), to find approximations for A_T and C_T. In this case, equation (20) and (21) are not satisfied exactly, and in order to find a solution we can solve equation (20), for example, in least squares sense. However, due to the consistency of calculating Γ_s, the estimates of A_T and C_T obtained in this way, will be consistent.

4.2. *Approximating the pair* (B_T, D)

In finding approximates for the pair (B_T, D) under the conditions specified in Theorem 1, we have a complementary theorem.

Theorem 3. Let the conditions of Theorem 1 hold, then:

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}Y_{1,s,N}(Q_1^N)^T = \lim_{N\to\infty}\frac{1}{\sqrt{N}}H_s(R_{11}^N), \tag{22}$$

and

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}Y_{s+1,s,N}(Q_2^N)^T = \lim_{N\to\infty}\frac{1}{\sqrt{N}}H_s(R_{22}^N). \tag{23}$$

Proof. See Appendix A.5.

From this Theorem, we can find an approximation of the Toeplitz matrix H_s in equation (7) as follows: from equations (22) and (23) and the RQ factorization in equation (9), we derive:

$$[R_{31}^N \quad R_{42}^N] = H_s[R_{11}^N \quad R_{22}^N] + O_N(\varepsilon).$$

Attributable to the white noise property of the input u_k, the right pseudo-inverse of the matrix $[R_{11}^N \quad R_{22}^N]$, denoted by $[R_{11}^N \quad R_{22}^N]^\dagger$, will always exist for sufficiently large N. Hence, the estimate of H_s, denoted by $\hat{H}_s$, equals:

$$\hat{H}_s = [R_{31}^N \quad R_{42}^N][R_{11}^N \quad R_{22}^N]^\dagger. \tag{24}$$

Under the conditions specified in Theorem 1, this estimate $\hat{H}_s$ is consistent.

Assume next that the matrix H_s is known exactly and the column space of Γ_s is equal to that of a known matrix U_n, then if we exploit the special Toeplitz structure of H_s, the pair of matrices (B_T, D) satisfies the following overdetermined set of equations:

$$\begin{pmatrix} I_l & 0 \\ 0 & U_n(1:\ell(s-1),:) \\ \hline I_\ell & 0 \\ 0 & U_n(1:\ell(s-2),:) \\ \hline \vdots & \vdots \\ \hline I_\ell & 0 \end{pmatrix}\begin{pmatrix} D \\ \hline B_T \end{pmatrix} = \begin{pmatrix} H_s(1:\ell s, 1:m_1) \\ \hline H_s(\ell+1:\ell s, m_1+1:2m_1) \\ \hline \vdots \\ \hline H_s(\ell(s-1)+1:\ell s, m_1(s-1)+1:m_1 s) \end{pmatrix}. \tag{25}$$

Caused by the consistency of the matrix $\hat{H}_s$ and the matrix $\hat{U}_n$, the pair of estimates obtained by solving equation (25) in least squares sense are also consistent for the conditions of Theorem 1.

Complementary to Theorem 2, we have the following theorem:

Theorem 4. Let the conditions of Theorem 2 hold, then:

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}Y_{1,s,N}(Q_1^{N'})^T = \lim_{N\to\infty}\left(\frac{1}{\sqrt{N}}\Gamma_s X_{1,N}(Q_1^{N'})^T + \frac{1}{\sqrt{N}}H_s(R_{21}^{N'})\right), \tag{26}$$

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}Y_{1,s,N}(Q_2^{N'})^T = \lim_{N\to\infty}\left(\frac{1}{\sqrt{N}}\Gamma_s X_{1,N}(Q_2^{N'})^T + \frac{1}{\sqrt{N}}H_s(R_{22}^{N'})\right), \tag{27}$$

and

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}Y_{s+1,s,N}(Q_1^{N'})^T = \lim_{N\to\infty}\left(\frac{1}{\sqrt{N}}\Gamma_s X_{s+1,N}(Q_1^{N'})^T + \frac{1}{\sqrt{N}}H_s(R_{11}^{N'})\right). \tag{28}$$

Proof. See Appendix A.6.

From Theorem 4, we can derive the key equations to approximate the pair (B_T, D). In order to see this, we use the RQ factorization in equation (16) to denote the equations in Theorem 4 as:

$$\frac{1}{\sqrt{N}}[R_{31}^{N'} \quad R_{32}^{N'} \quad R_{41}^{N'}] = \Gamma_s\frac{1}{\sqrt{N}}[X_{1,N}(Q_1^{N'})^T \quad X_{1,N}(Q_2^{N'})^T \quad X_{s+1,N}(Q_1^{N'})^T] + H_s\frac{1}{\sqrt{N}}[R_{21}^{N'} \quad R_{22}^{N'} \quad R_{11}^{N'}] + O_N(\varepsilon). \tag{29}$$

When the column spaces of the matrix Γ_s and its orthogonal complement are respectively equal to that of U_n and $U_n^{\perp}$, then we can multiply equation (29) on the left by $(U_n^{\perp})^T$, and obtain:

$$(U_n^{\perp})^T \frac{1}{\sqrt{N}}[R_{31}^{N'} \quad R_{32}^{N'} \quad R_{41}^{N'}]$$

$$= (U_n^{\perp})^T H_s \frac{1}{\sqrt{N}}[R_{21}^{N'} \quad R_{22}^{N'} \quad R_{11}^{N'}] + O_N(\varepsilon). \quad (30)$$

Caused by the sufficiently persistence of excitation of the input, the matrix $[R_{21}^{N'} \quad R_{22}^{N'} \quad R_{11}^{N'}]$ has a right pseudo-inverse. Multiplying equation (30) on the right by this inverse, yields:

$$(U_n^{\perp})^T \frac{1}{\sqrt{N}}[R_{31}^{N'} \quad R_{32}^{N'} \quad R_{41}^{N'}]$$

$$\left(\frac{1}{\sqrt{N}}[R_{21}^{N'} \quad R_{22}^{N'} \quad R_{11}^{N'}]\right)^{\dagger} = (U_n^{\perp})^T H_s + O_N(\varepsilon).$$

If we denote the left hand side of this equation by $\Xi \in R^{(\ell s-n)\times m_1 s}$, then this equation can be written as:

$$\Xi = (U_n^{\perp})^T H_s + O_N(\varepsilon). \quad (31)$$

Apart from the error term $O_N(\varepsilon)$, which vanishes for $N \to \infty$, equation (31) is identical to the corresponding equation in the ordinary MOESP scheme, from which the pair (B_T, D) is computed (see the proof of Theorem 5 in Verhaegen and Dewilde (1992a)). The calculation of the latter quantities is done via the solution of the overdetermined set of equations given by equation (45) in Verhaegen and Dewilde (1992a) with m_1 substituted for m.

4.3. *Two identification schemes*

From the previous theorems, we can derive two schemes that provide consistent estimates for the identification problem formulated in the introduction.

Based on Theorems 1 and 3, we formulate a first algorithm.

Algorithm 1.

Given:

—an estimate of the underlying system order n. The latter estimate could be obtained from the SVD determined later in Step 3. As such, the order n no longer acts as an input to the algorithm, but is determined during its operation. In practice, the estimation of the system order amounts to the partitioning of the computed singular values into 'larger' and 'smaller' ones. The number of 'large' singular values is then an estimate of the system order n. It should be noted that deciding on which singular values are 'large' and which are 'small' may not be trivial. For a more elaborate discussion on estimating the system order, we refer to Verhaegen (1993a), where this topic is discussed in more detail for related SMI schemes.

—a dimension parameter s, satisfying:

$$s > n.$$

—the input and output sequences:

$$[u_1\, u_2 \cdots u_{N+2s-1}] \quad \text{and} \quad [y_1\, y_2 \cdots y_{N+2s-1}]$$

for $N \gg s$ and the input a realization of a zero-mean white noise sequence.

Do the following:

Step 1. Construct the Hankel matrices $U_{1,s,N}$, $U_{s+1,s,N}$, $Y_{1,s,N}$ and $Y_{s+1,s,N}$.

Step 2. Do a data compression via an RQ factorization as specified in equation (19) without accumulating the orthogonal transformations required.

Step 3. Compute the SVD of the matrix $[R_{41}^{N} \quad R_{43}^{N}]$ as given in equation (14).

Step 4. Solve the set of equations, equations (20) and (21) and equation (25) using the estimate of the matrix U_n computed in Step 3 and the estimated Toeplitz matrix $\hat{H}_s$ computed in equation (24).

Remark 2. Even when $w_k \equiv 0$ and $v_k \equiv 0$, Algorithm I will only yield consistent estimates. This is because this algorithm still relies on Lemma 1. □

Based on Theorems 2 and 4, we formulate a second algorithm. This algorithm is indicated by the ordinary MOESP scheme with instrumental variables constructed from past input and past output measurements, in short the PO scheme, for reasons explained in the next section.

Algorithm 2. The PO scheme.

Given:

—the same items as in Algorithm 1 are required, however, now the input sequence only has to be sufficiently persistently exciting.

Do the following:

Step 1. Construct the Hankel matrices $U_{1,s,N}$, $U_{s+1,s,N}$, $Y_{1,s,N}$ and $Y_{s+1,s,N}$.

Step 2. Do a data compression via an RQ factorization as specified in equation (16) again without accumulating the orthogonal transformations required.

Step 3. Compute the SVD of the matrix $[R_{42}^{N'} \quad R_{43}^{N'}]$ as given in equation (19).

Step 4. Solve the set of equations equations (20) and (21) and equation (45) of Verhaegen and Dewilde (1992a) using the estimate of the matrix

U_n' and $U_n^{\perp'}$ defined in Step 3 and the estimated matrix Ξ computed in equation (31).

Remark 3. This algorithm gives exact result for finite length data sequences when $w_k \equiv 0$ and $v_k \equiv 0$. □

Remark 4. Experimental experience with the schemes derived in Verhaegen and Dewilde (1992a), Verhaegen and Dewilde (1992b) and Verhaegen (1993a) and the PO algorithm, has shown that the construction of the overdetermined set of equations in equation (45) of Verhaegen and Dewilde (1992a) is very time consuming. However, exploiting the special structure of the matrices in the underbraced term in equation (45) of Verhaegen and Dewilde (1992a) some computational savings can be obtained. When doing the matrix multiplication in the underbraced term in equation (45) of Verhaegen and Dewilde (1992a) straightforwardly, the number of multiplications is equal to:

$$s^3\ell^2(\ell+n) - s^2\ell n(\ell+n).$$

However, taking into account all the zero block matrices and the identity matrix in equation (45) of Verhaegen and Dewilde (1992a), the computational complexity becomes:

$$\tfrac{1}{2}(s^3\ell^2 n - s^2\ell n(\ell+n) + s\ell n^2).$$

For example, for particular values of s, ℓ and n equal to 20, 2 and 4, respectively, this results in a speed-up of more than a factor 3. □

5. THE INSTRUMENTAL VARIABLES OF THE PO SCHEME

The special organization of the RQ factorization in equation (16) makes the PO scheme a special variant of the ordinary MOESP scheme extended with instrumental variables (Verhaegen, 1993a).

In order to show this, let us briefly recall the basic operation of instrumental variables within the MOSEP framework. When an instrumental variable sequence has been chosen, we use this sequence to construct a (structured) matrix W_N. This matrix has N columns and a specific number of rows, depending on the chosen instrumental variable. Next we consider the following RQ factorization:

$$\begin{pmatrix} U_{s+1,s,N} \\ W_N \\ Y_{s+1,s,N} \end{pmatrix} = \begin{pmatrix} L_{11}^N & & \\ L_{21}^N & L_{22}^N & \\ L_{31}^N & L_{32}^N & L_{33}^N \end{pmatrix} \begin{pmatrix} Q_1^N \\ Q_2^N \\ Q_3^N \end{pmatrix}. \quad (32)$$

In a way similar to that in Theorems 1 and 2, it is shown in Verhaegen (1993a) that when the input to a deterministic FDLTI system is zero mean white noise and its output is perturbed by an additive zero-mean stochastic process of arbitrary colour which is statistically independent of the instrumental variable sequence, then the following relationship holds:

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} L_{32}^N = \lim_{N\to\infty} \frac{1}{N} \Gamma_s X_{s+1} W_N^T \left(\frac{1}{\sqrt{N}} L_{22}^N\right)^{-T}. \quad (33)$$

Based on this relationship, an approximation of the column space of the extended observability matrix Γ_s can be retrieved from an SVD of the matrix L_{32}^N. In Verhaegen (1993a), two types of instrumental variable matrices W_N are considered:

- $W_N = U_{1,s,N}$. This scheme was indicated in Verhaegen (1993a) as the PI, standing for Past Input, scheme.
- W_N equal to a reconstruction of the time sequence of the state vector of the deterministic system. For a discussion on how to reconstruct this state sequence, we refer to Verhaegen (1993a). This scheme was indicated by the RS, standing for Reconstructed State, scheme.

By comparing the RQ factorization in equation (32) with that used in Algorithm 2, see equation (16), we immediately observe that

$$W_N = \begin{pmatrix} U_{1,s,N} \\ Y_{1,s,N} \end{pmatrix}$$

and that Algorithm 2 indeed is a special variant of the instrumental variable approach proposed in Verhaegen (1993a).

To gain more insight in which part of this choice of W_N is active, namely that part of W_N that remains present in limit relationship like equation (33), we state our final theorem.

Theorem 5. Let the process noise w_k and the measurement noise v_k of the system (1–2) be discrete zero-mean white noise, independent of the input u_j for all k, j and of the initial state x_0, let the input u_k to the system (1–2) be discrete zero-mean white noise, and let the RQ factorization in equation (16) be given, then:

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} Y_{s+1,s,N}(Q_2^{N'})^T = \lim_{N\to\infty} \frac{1}{N} \Gamma_s X_{s+1,N} U_{1,s,N}^T \left(\frac{1}{\sqrt{N}} R_{22}^{N'}\right)^{-T}, \quad (34)$$

and

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} Y_{s+1,s,N}(Q_3^{N'})^T$$
$$= \lim_{N\to\infty} \Gamma_s \frac{1}{N} X_{s+1,N}(X_{1,N}^T \Gamma_s^T + W_{1,s,N}^T E_s^T)$$
$$\times \left(\frac{1}{\sqrt{N}} R_{33}^{N'}\right)^{-T}. \tag{35}$$

Proof. See Appendix A.7.

From equations (34) and (35), we conclude that three parts present in the instrumental variable matrix W_N are active in retrieving the column space of Γ_s. These are the matrices $U_{1,s,N}$, $X_{1,N}$ and $W_{1,s,N}$. It is important to note that, contrary to the RS scheme, the latter two matrices are obtained without the need to reconstruct.

In analogy with the choice of acronyms for the above mentioned instrumental variable extensions of the ordinary MOESP scheme, Algorithm 2 should be indicated as the PIPO scheme. However, we prefer the use of the acronym PO, since the above three active quantities are present in the Past Output data Hankel matrix $Y_{1,s,N}$.

Using the sensitivity analysis framework set-up in Verhaegen (1993a), it is possible to compare the numerical sensitivity of the PO scheme with both the RS and the PI schemes. The conclusion of such a sensitivity analysis is that from these three instrumental variable approaches, the PO scheme will lead to the most accurate approximation of the column space of Γ_s for the class of identification problems stated in the introduction.

Since a formal proof of this conclusion can be given analogously to the proofs establishing the numerical sensitivity of the PI and RS scheme, given in Verhaegen (1993a), we restrict ourselves to sketching an outline of such a proof. Let us assume that the conditions in Theorem 5 hold, then the result of this theorem forms the main ingredient. Equations (34) and (35) of this theorem show that the singular values of the matrix,

$$\frac{1}{N}\Gamma_s \left[\underbrace{X_{s+1,N} U_{1,s,N}^T \left(\frac{1}{\sqrt{N}} R_{22}^{N'}\right)^{-T}}_{1} \right.$$
$$\left. \underbrace{(X_{s+1,N} X_{1,N}^T \Gamma_s^T}_{2} + X_{s+1,N} W_{1,s,N}^T E_s^T)\left(\frac{1}{\sqrt{N}} R_{33}^{N'}\right)^{-T} \right] \tag{36}$$

are always greater than or equal to the singular values of the matrix only containing the underbraced Term 1 or 2 of the term in equation (36) between square brackets. When only Term 1 is retained, we obtain the condition present in the PI scheme. The other case represents the condition in the RS scheme relying on a perfect reconstruction of the state quantities of the system described by equation (1).

For finite N it is only possible to get approximations of the quantity in equation (36) or one of its variants corresponding to the PI or RS scheme. However, if we assume that the additive errors to these approximations are of the same order of magnitude in all three cases, then the above insight into the singular values directly shows, via an application of a basic result on the sensitivity analysis of the SVD, see e.g. Stewart (1973), that the stated conclusion holds. In the simulation study, in Section 6.1, we illustrate the assumption used in coming to this conclusion.

A consequence of the previous conclusion is that the PO scheme will lead to more accurate estimates of the pair $[A_T, C_T]$.

Finally, we mention that in the PO scheme, the instrumental variable matrix is not only used in the estimation of the column space of Γ_s, as is the case with the other two instrumental variable approaches. From equation (30) we observe that the part of the past output data Hankel matrix $Y_{1,s,N}$ corresponding to the submatrices $R_{31}^{N'}$ and $R_{32}^{N'}$ is used in the calculation of the pair $[B_T, D]$. Caused by this more effective use of the output (compared to the case when only using the part $R_{41}^{N'}$ related to the future data Hankel matrix) a better estimation of the pair $[B_T, D]$ might be expected.

6. SIMULATION STUDY

In this section, we report the results of two simulation studies to highlight the usefulness of the algorithms presented in this paper. In these studies we compare the presented algorithms with other members of the MOESP class of algorithms. In the first example, a comparison is made with the PI scheme, presented in Verhaegen (1993a). The latter scheme yields consistent estimates when in Fig. 1, $w_k \equiv 0$ and v_k is zero-mean but of arbitrary colour. In the second example, the realistic circumstances of an aircraft flying through gusty wind are simulated. In this example, a comparison is made with the RS scheme, that yielded in Verhaegen (1993a) the most reliable and unbiased results.

6.1. *A simple first-order system*

A first simple example that already gives some interesting insights into the performances of the

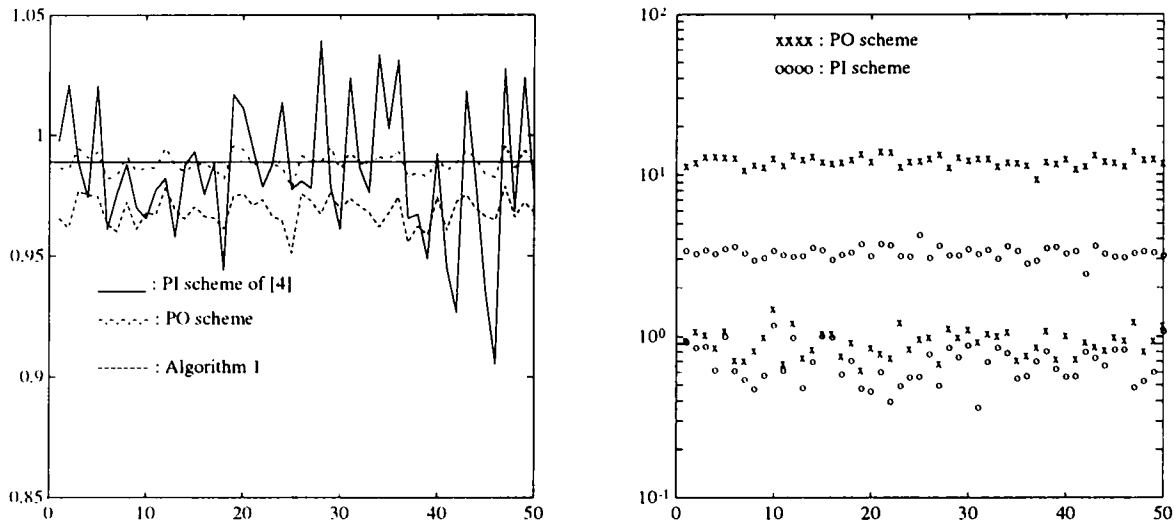

FIG. 2. Estimated A matrix by Algorithm 1, the PO and PI (lhs) and the two largest singular values calculated in the PO and PI schemes (rhs) for the experiment presented in Section 6.1.

presented algorithms is treated in this subsection.

The true system to be identified is a single input, single output (SISO) model given as:

$$x_{k+1} = 0.9890x_k + 1.8805u_k - 0.1502w_k, \quad (37)$$

$$y_k = 0.8725x_k - 2.0895u_k + 2.5894w_k. \quad (38)$$

The input sequence u_k and the process noise w_k are independent zero-mean white noise sequences of unit variance. The length of the observed input and output sequences is 200 and a Monte-Carlo experiment is conducted whereby in each run we generate another process noise sequence w_k. A total number of 50 runs are performed, each run the dimension index s was taken equal to six in the different identification schemes. The computations are performed within the Matlab package (Moler *et al.*, 1987). As a result of this experiment, we focus on the estimation of the transition matrix A. The estimates are plotted in the left-hand side (lhs) of Fig. 2. From this figure, we observe that Algorithm 1 leads to biased estimates. This indicates that the consistency results of Theorem 1 are not very robust and require a large number of samples. With the present example, more or less the same type of unbiased estimates were obtained with Algorithm 1 as with the PO scheme when N was chosen equal to 1000. The PO scheme on the other hand does not suffer from this deficiency. These results are compared with those obtained with the PI scheme. Based on this comparison, see the lhs shown in Fig. 2, we conclude that both methods discussed in this paper yield estimates with a reduced variance of a factor ≈ 25. This supports the conclusion stated in the previous section.

In the right-hand side (rhs) of Fig. 2 we plot the two largest singular values of the set computed by the PO and the PI scheme during the 50 different runs. Since the other singular values in both cases are of the same order of magnitude as the second largest one, for the sake of clarity, we do not display those. From these singular value plots a number of things can be observed. First, since there is a clear gap between the first singular value and the rest, they allow us to correctly estimate that the order is 1. Second, this gap is more striking in the set of singular values corresponding to the PO scheme, indicated by *xxxx* in Fig. 2, then in those corresponding to the PI scheme, indicated by *oooo* in Fig. 2. Third, we see that the 'small' singular values are of the same order of magnitude in both the PI and the PO scheme. This then illustrates the assumption made in coming to that conclusion.

6.2. *Identification of the aircraft dynamics when flying through gusty wind*

This example, which was treated in Verhaegen (1993a), is of the class of systems analyzed in this paper. Further, since it was indicated to be a critical system in Verhaegen (1993a), we reanalyze the example to see what improvements may be obtained using the schemes presented in this paper.

The mathematical model. The particular aircraft analyzed in this experiment is an F-8 aircraft and the numerical data describing its dynamics is taken from Elliott (1977). The continuous time model that describes the linearized longitudinal motion of the aircraft hit by a vertical gusty wind at an altitude of 20,000 ft, an airspeed of 620 ft s^{-1} and an angle of attack $\alpha_0 = 0.078$ rad

is:

$$\frac{d}{dt}\begin{pmatrix} q \\ u \\ \alpha \\ \theta \end{pmatrix} = \begin{pmatrix} -0.49 & 0.00005 & -4.8 & 0 \\ 0 & -0.015 & -14.0 & -32.2 \\ 1.0 & -0.00019 & -0.84 & 0 \\ 1.0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} q \\ u \\ \alpha \\ \theta \end{pmatrix} + \begin{pmatrix} -8.7 \\ -1.1 \\ -0.11 \\ 0 \end{pmatrix} \delta_e + \begin{pmatrix} -4.8 \\ -14.0 \\ -0.84 \\ 0 \end{pmatrix} \alpha_g \quad (39)$$

Here q is the rate of pitch, u the horizontal component of the airspeed, α the angle of attack, θ the pitch angle, δ_e the measured elevator deflection angle (deterministic) and α_g the unmeasurable scaled vertical gust velocity. Zero-mean white noise with standard deviation equal to 0.05 and 0.2 affect the output measurements q and u, respectively.

In the simulation, use is made of a discrete version of the model in equation (39) for a discrete period of $\Delta t = 0.05$ s and a zero-mean white noise sequence with standard deviation $\sigma_\delta = 0.1$ for the δ_e.

The vertical gust velocity α_g is simply taken to be zero-mean white noise with standard deviation σ_w equal to 0.2. Again we perform a similar Monte-Carlo simulation study as in the previous experiment and plot the eigenvalues of the estimated transition matrix. A total number of 100 runs is performed. The dimension parameter s is kept equal to 20. The results of the RS scheme, displayed in the rhs of our final figure, clearly demonstrate that the latter scheme produces very sensitive results. Therefore, this example may be considered as one that cannot be analyzed with the schemes presented in Verhaegen (1993a). However, the lhs of Fig. 3 shows that the PO scheme yielded unbiased estimates with a small(er) variance. Again these observations can be supported by inspecting the corresponding singular values. For the sake of brevity the latter quantities are not presented here.

Finally, we mention that Algorithm 1 yielded the same results as the PO scheme. This confirms that Theorem 1 holds when the length of the data batches is 'large'.

7. CONCLUDING REMARKS

In this paper, we have derived two algorithms to identify a MIMO state space model from perturbed input–output data. The presented algorithms share the algorithmic structure of the MOESP class of algorithms, recently presented in Verhaegen and Dewilde (1992a, b) and Verhaegen (1993a). This is because the main algorithmic steps of the algorithms presented in this paper, are identical to those of the mentioned papers. The relationship with the schemes presented in Verhaegen (1993a) based on instrumental variables is explicitly outlined in this paper for the second new algorithm. The latter scheme is indicated by the PO, standing for Past Input (instrumental variables). This outline leads to the conclusion that the PO scheme allows to approximate the key quantity in the MOESP class of algorithms, namely the column space of the extended observability matrix of the system to be identified, more accurately than the instrumental variable schemes derived in Verhaegen (1993a). As a consequence, the system matrices derived from this quantity are determined with greater precision. This conclusion is supported in the simulation analysis section.

As outlined in the identification problem

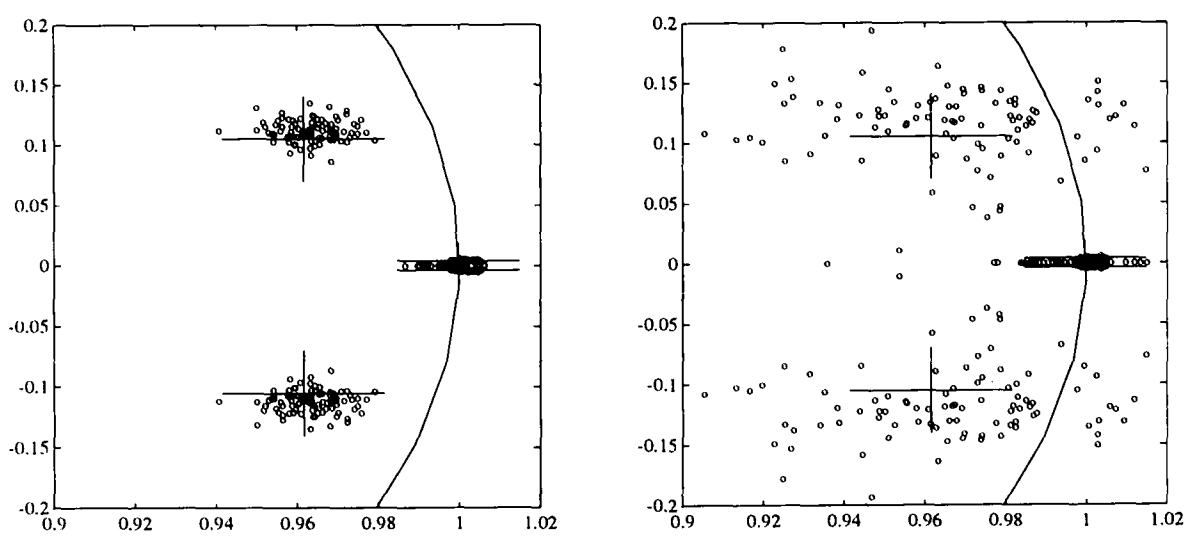

FIG. 3. Poles of the estimated fourth-order systems with the PO scheme (lhs) and the RS scheme for the experiment presented in Section 6.2.

stated in the introduction, we restrict the analysis in this paper to the identification of the deterministic part of the system model. In Overschee and De Moor (1992), the problem of identifying the stochastic part is treated also. Preliminary results on this subject within the framework presented in this paper are reported in Verhaegen (1993b).

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APPENDIX A: PROOFS OF THE LEMMAS AND THEOREMS

A.1. Proof of Lemma 1

From the white noise property of u_k in Definition 1 and the representation in equation (6), we conclude that:

$$\frac{1}{N} U_{1,2s,N} U_{1,2s,N}^T = \sigma_u^2 I_{2m_1 s} + O_N(\varepsilon).$$

Using the RQ factorization in equation (9), the above equation is equal to:

$$\begin{pmatrix} \frac{1}{N} R_{11}^N (R_{11}^N)^T & \frac{1}{N} R_{11}^N (R_{21}^N)^T \\ \frac{1}{N} R_{21}^N (R_{11}^N)^T & \frac{1}{N}(R_{21}^N (R_{21}^N)^T + R_{22}^N (R_{22}^N)^T) \end{pmatrix} = \begin{pmatrix} \sigma_u^2 I_{m_1 s} + O_N(\varepsilon) & O_N(\varepsilon) \\ O_N(\varepsilon) & \sigma_u^2 I_{m_1 s} + O_N(\varepsilon) \end{pmatrix}.$$

Hence,

$$\frac{1}{N} R_{21}^N (R_{11}^N)^T = O_N(\varepsilon) \quad \text{and} \quad \frac{1}{N} R_{11}^N (R_{11}^N)^T = \sigma_u^2 I_{m_1 s} + O_N(\varepsilon).$$

From the right-hand side of this equation, we conclude that for sufficiently large N the matrix $(1/\sqrt{N} R_{11}^N)$ remains invertible and therefore the left-hand side becomes:

$$\frac{1}{\sqrt{N}} R_{21}^N = O_N(\varepsilon)$$

which concludes the proof of the lemma. □

A.2. Proof of Lemma 3

The proof of both equations in Lemma 3 is very similar. Therefore, we only prove the right-hand side equation.

Using equation (7) for $j = 1$, we find that:

$$\frac{1}{N} Y_{1,s,N} W_{s+1,s,N}^T = \underbrace{\Gamma_s X_{1,N} W_{s+1,s,N}^T}_{(i)} + \underbrace{H_s U_{1,s,N} W_{s+1,s,N}^T}_{(ii)} + \underbrace{E_s W_{1,s,N} W_{s+1,s,N}^T}_{(iii)} + \underbrace{V_{1,s,N} W_{s+1,s,N}^T}_{(iv)}.$$

By Lemma 2, the term (i) is $O_N(\varepsilon)$. Attributable to the statistical independency of u_k and w_i for $\forall k, i$, the term (ii) is also $O_N(\varepsilon)$ and the same is true for the terms (iii) and (iv) caused by the white noise property of w_k and v_k, as specified by equation (3). Hence,

$$\frac{1}{N} Y_{1,s,N} W_{s+1,s,N}^T = O_N(\varepsilon). \tag{A.1}$$

Using the expression for $U_{1,s,N}$ in equation (9) and the independency of u_k and w_i for $\forall k, i$, we can write:

$$\frac{1}{N} R_{11}^N Q_1^N W_{s+1,s,N}^T = \frac{1}{N} U_{1,s,N} W_{s+1,s,N}^T = O_N(\varepsilon).$$

Caused by the persistency of excitation of the input, the matrix R_{11}^N is invertible. Therefore,

$$\frac{1}{\sqrt{N}} Q_1^N W_{s+1,s,N}^T = O_N(\varepsilon). \tag{A.2}$$

Using this expression, and again the independency of u_k and w_i for $\forall k, i$, stated as:

$$\frac{1}{N} U_{s+1,s,N} W_{s+1,s,N}^T = O_N(\varepsilon),$$

we can similarly derive that:

$$\frac{1}{\sqrt{N}} Q_2^N W_{s+1,s,N}^T = O_N(\varepsilon). \tag{A.3}$$

Finally, with the expression for $Y_{1,s,N}$, given in equation (9), the term $(1/N) Y_{1,s,N} W_{s+1,s,N}^T$ can be written as:

$$\frac{1}{N} Y_{1,s,N} W_{s+1,s,N}^T = \frac{1}{\sqrt{N}} R_{31}^N \frac{1}{\sqrt{N}} Q_1^N W_{s+1,s,N}^T + \frac{1}{\sqrt{N}} R_{32}^N \frac{1}{\sqrt{N}} Q_2^N W_{s+1,s,N}^T + \frac{1}{\sqrt{N}} R_{33}^N \frac{1}{\sqrt{N}} Q_3^N W_{s+1,s,N}^T$$

Using equations (A.1–A.3), we obtain:

$$\frac{1}{\sqrt{N}} R_{33}^N \frac{1}{\sqrt{N}} Q_3^N W_{s+1,s,N}^T = O_N(\varepsilon).$$

Attributable to the white noise property of w_i, v_i and the independency of u_k and w_i, v_i for $\forall k$, i, the matrix R_{33}^N is invertible and the result of the lemma has been proved. □

A.3. Proof of Theorem 1

Using equation (7) for $j = s + 1$ and the RQ factorization in equation (9), we find that:

$$\frac{1}{\sqrt{N}} Y_{s+1,s,N}(Q_1^N)^T = \frac{1}{\sqrt{N}} \Gamma_s X_{s+1,N}(Q_1^N)^T + H_s \frac{1}{\sqrt{N}} R_{21}^N + E_s \frac{1}{\sqrt{N}} W_{s+1,s,N}(Q_1^N)^T + \frac{1}{\sqrt{N}} V_{s+1,s,N}(Q_1^N)^T.$$

Caused by the independency of u_k and v_i for $\forall k$, i, one can show, similarly to equation (A.2) that:

$$\frac{1}{\sqrt{N}} V_{s+1,s,N}(Q_1^N)^T = O_N(\varepsilon).$$

Therefore, by Lemma 1 and equation (A.2), we obtain equation (10) of the theorem. Again using equation (7) for $j = s + 1$ and the RQ factorization in equation (9), we find that:

$$\frac{1}{\sqrt{N}} Y_{s+1,s,N}(Q_3^N)^T = \frac{1}{\sqrt{N}} \Gamma_s X_{s+1,N}(Q_3^N)^T + E_s \frac{1}{\sqrt{N}} W_{s+1,s,N}(Q_3^N)^T + \frac{1}{\sqrt{N}} V_{s+1,s,N}(Q_3^N)^T.$$

By Lemma 3, the second and third term in the rhs of this equation is $O_N(\varepsilon)$. Therefore, also equation (11) is proved□

A.4. Proof of Theorem 2

Now using equation (7) for $j = s + 1$ and the RQ factorization in equation (16), we find that:

$$\frac{1}{\sqrt{N}} Y_{s+1,s,N}(Q_2^{N'})^T = \frac{1}{\sqrt{N}} \Gamma_s X_{s+1,N}(Q_2^{N'})^T + E_s \frac{1}{\sqrt{N}} W_{s+1,s,N}(Q_2^{N'})^T + \frac{1}{\sqrt{N}} V_{s+1,s,N}(Q_2^{N'})^T.$$

As in the proof of Theorem 1, we use the independency of u_k and w_i resp. v_i for $\forall k$, i, to show that:

$$\frac{1}{\sqrt{N}} W_{s+1,s,N}(Q_2^{N'})^T = O_N(\varepsilon)$$

and

$$\frac{1}{\sqrt{N}} V_{s+1,s,N}(Q_2^{N'})^T = O_N(\varepsilon).$$

Therefore, we obtain equation (17) of the theorem without making use of Lemma 1. When making use of Remark 1, the proof of equation (18) becomes very similar to the proof of equation (11) in Theorem 1. □

A.5. Proof of Theorem 3

Using equation (7) for $j = 1$ and the RQ factorization in equation (9), we find that:

$$\frac{1}{\sqrt{N}} Y_{1,s,N}(Q_1^N)^T = \frac{1}{\sqrt{N}} \underbrace{\Gamma_s X_{1,N}(Q_1^N)^T}_{(i)} + \frac{1}{\sqrt{N}} H_s R_{11}^N + \frac{1}{\sqrt{N}} \underbrace{(E_s W_{1,s,N}(Q_1^N)^T + V_{1,s,N}(Q_1^N)^T)}_{(iii)}.$$

When we substitute u_k for w_k in Lemma 2, this Lemma then shows that the (i) term in the above equations is $O_N(\varepsilon)$. Using the independency of u_k and w_i, v_i for $\forall k$, i, one can easily show, as in the proof of Lemma 3, see e.g. equation (A.2), that:

$$\frac{1}{\sqrt{N}} W_{1,s,N}(Q_1^{N'})^T = O_N(\varepsilon)$$

and

$$\frac{1}{\sqrt{N}} V_{1,s,N}(Q_1^{N'})^T = O_n(\varepsilon).$$

Therefore, the (ii) term is $O_N(\varepsilon)$ and equation (22) is proved.

By similar arguments, equation (23) can be proved. □

A.6. Proof of Theorem 4

The proof of each relationship in Theorem 4 is very similar. Therefore, we only prove equation (26). Using equation (7) for $j = 1$ and the RQ factorization in equation (16), we find that:

$$\frac{1}{\sqrt{N}} Y_{1,s,N}(Q_1^{N'})^T = \frac{1}{\sqrt{N}} \Gamma_s X_{1,N}(Q_1^{N'})^T + \frac{1}{\sqrt{N}} H_s R_{21}^{N'} + \frac{1}{\sqrt{N}} \underbrace{(E_s W_{1,s,N}(Q_1^{N'})^T + V_{1,s,N}(Q_1^{N'})^T)}_{(i)}.$$

By the independency of u_k and w_i, v_i for $\forall k$, i, the (i) term is $O_N(\varepsilon)$ and therefore equation (26) is proved. □

A.7. Proof of Theorem 5

We first prove equation (34). We start with equation (7) for $j = s + 1$. With the RQ factorization given in equation (16), we obtain:

$$\frac{1}{N} Y_{s+1,s,N}(Q_2^{N'})^T (R_{22}^{N'})^T = \frac{1}{N} \Gamma_s X_{s+1,N}(Q_2^{N'})^T (R_{22}^{N'})^T + E_s \frac{1}{\sqrt{N}} W_{s+1,s,N}(Q_2^{N'})^T \left(\frac{1}{\sqrt{N}} R_{22}^{N'}\right)^T + \frac{1}{\sqrt{N}} V_{s+1,s,N}(Q_2^{N'})^T \left(\frac{1}{\sqrt{N}} R_{22}^{N'}\right)^T.$$

Caused by the independency of u_k and w_i for $\forall k$, i, the underbraced terms are again $O_N(\varepsilon)$. Therefore,

$$\frac{1}{\sqrt{N}} Y_{s+1,s,N}(Q_2^{N'})^T \left(\frac{1}{\sqrt{N}} R_{22}^{N'}\right)^T = \frac{1}{\sqrt{N}} \Gamma_s X_{s+1,N}(Q_2^{N'})^T \left(\frac{1}{\sqrt{N}} R_{22}^{N'}\right)^T + O_N(\varepsilon).$$

By Lemma 2, and the RQ factorization in equation (16), the following holds:

$$\frac{1}{\sqrt{N}} X_{s+1,N}(Q_1^{N'})^T = O_N(\varepsilon).$$

Hence,

$$\frac{1}{\sqrt{N}} Y_{s+1,s,N}(Q_2^{N'})^T \left(\frac{1}{\sqrt{N}} R_{22}^{N'}\right)^T = \frac{1}{\sqrt{N}} \Gamma_s X_{s+1,N} \left[(Q_2^{N'})^T \left(\frac{1}{\sqrt{N}} R_{22}^{N'}\right)^T + (Q_1^{N'})^T \left(\frac{1}{\sqrt{N}} R_{21}^{N'}\right)^T\right] + O_N(\varepsilon)$$

and the proof of the first part of the Theorem is completed. Using equation (7) for $j = 1$ and the RQ factorization in equation (16), we obtain:

$$Y_{1,s,N} = \Gamma_s X_{1,N} + H_s U_{1,s,N} + E_s W_{1,s,N} + V_{1,s,N} = R_{31}^{N'} Q_1^{N'} + R_{32}^{N'} Q_2^{N'} + R_{33}^{N'} Q_3^{N'}. \quad \text{(A.4)}$$

From equation (29), we deduce:

$$\frac{1}{\sqrt{N}} \Gamma_s X_{1,N}(Q_1^{N'})^T + \frac{1}{\sqrt{N}} H_s R_{21}^{N'} = \frac{1}{\sqrt{N}} R_{31}^{N'} + O_N(\varepsilon)$$

and

$$\frac{1}{\sqrt{N}} \Gamma_s X_{1,N}(Q_2^{N'})^T + \frac{1}{\sqrt{N}} H_s R_{22}^{N'} = \frac{1}{\sqrt{N}} R_{32}^{N'} + O_N(\varepsilon).$$

Inserting these expressions in equation (A.4) yields:

$$\frac{1}{\sqrt{N}}(\Gamma_s X_{1,N}(I-(Q_1^{N'})^T Q_1^{N'}-(Q_2^{N'})^T Q_2^{N'})+E_s W_{1,s,N}+V_{1,s,N})$$
$$=\frac{1}{\sqrt{N}}R_{33}^{N'}Q_3^{N'}+O_N(\varepsilon)Q_1^{N'}+O_N(\varepsilon)Q_2^{N'}. \quad \text{(A.5)}$$

By Lemma 2, we have that:

$$\frac{1}{N}X_{1,N}U_{s+1,s,N}^T=O_N(\varepsilon) \quad \text{and} \quad \frac{1}{N}X_{1,N}U_{1,s,N}^T=O_N(\varepsilon).$$

Hence, using the RQ factorization in equation (16) and the white noise property of u_k, we derive from these two equations that:

$$\frac{1}{\sqrt{N}}X_{1,N}(Q_1^{N'})^T=O_N(\varepsilon)$$

and

$$\frac{1}{\sqrt{N}}X_{1,N}(Q_2^{N'})^T=O_N(\varepsilon).$$

Hence, equation (A.5) reduces to:

$$\frac{1}{\sqrt{N}}R_{33}^{N'}Q_3^{N'}=\frac{1}{\sqrt{N}}(\Gamma_s X_{1,N}+E_s W_{1,s,N}+V_{1,s,N})$$
$$+O_N(\varepsilon)Q_1^{N'}+O_N(\varepsilon)Q_2^{N'}. \quad \text{(A.6)}$$

Next, consider the expression $(1/N)Y_{s+1,s,N}(Q_3^{N'})^T(R_{33}^{N'})^T$. Using equation (7) for $j=s+1$ and equation (A.6), this can be expressed as:

$$\frac{1}{N}Y_{s+1,s,N}(Q_3^{N'})^T(R_{33}^{N'})^T=\frac{1}{N}(\Gamma_s X_{s+1,N}+E_s W_{s+1,s,N}$$
$$+V_{s+1,s,N})(X_{1,N}^T\Gamma_s^T+W_{1,s,N}^T E_s^T+V_s^T+V_{1,s,N}^T)+O_N(\varepsilon).$$

Using the white noise property of w_k and v_k and the independency of x_k and v_i for $\forall k, i$, the rhs simplifies to:

$$=\Gamma_s\frac{1}{N}X_{s+1,N}X_{1,N}^T\Gamma_s^T+E_s\underbrace{\frac{1}{N}W_{s+1,s,N}X_{1,N}^T\Gamma_s^T}_{(i)}$$
$$+\Gamma_s\frac{1}{N}X_{s+1,N}W_{1,s,N}^T E_s^T+O_N(\varepsilon).$$

By Lemma 2, the (i) term is $O_N(\varepsilon)$ and hence equation (35) is proved. □